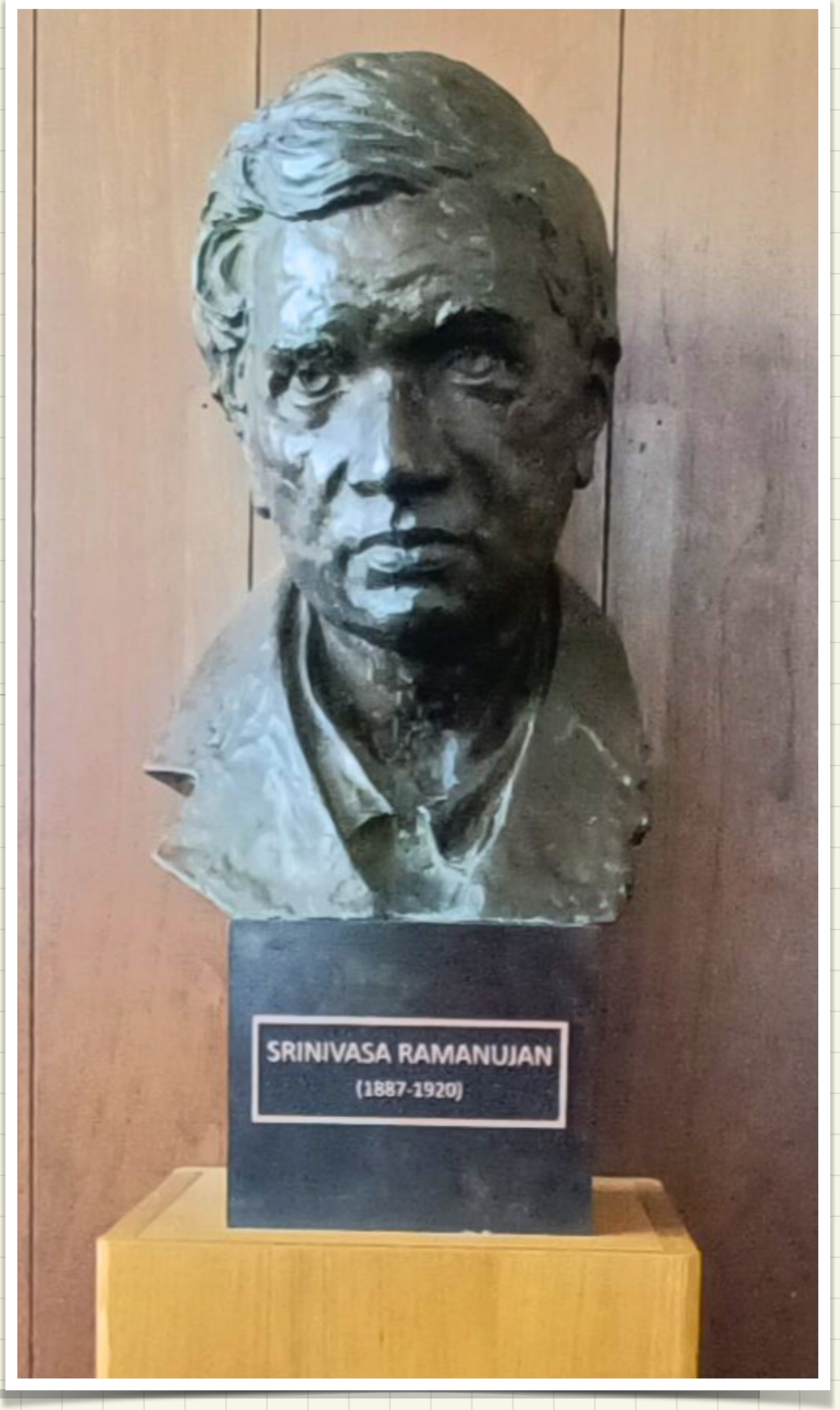
RAMANUJAN EXPLAINED

Lectures by Gaurav Bhatnagar

Lecture 1:
How to discover the Rogers-Ramanujan identities



THE ROGERS-RAMANUJAN IDENTITIES

$$1+\frac{q}{(1-q)}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots$$
$$=\frac{1}{(1-q)(1-q^6)(1-q^{11})(1-q^{16})\cdots}\times\frac{1}{(1-q^4)(1-q^9)(1-q^{14})\cdots}$$

$$1+\frac{q^2}{(1-q)}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots$$
$$=\frac{1}{(1-q^2)(1-q^7)(1-q^{12})(1-q^{17})\cdots}\times\frac{1}{(1-q^3)(1-q^8)(1-q^{13})\cdots}$$

> “It would be difficult to find more beautiful formulae than the Rogers-Ramanujan identities…”
>
> *–G.H. Hardy*

$$\sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\cdots(1-q^k)} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}$$

How do we make sense of this?

1 - Formal power series in q ✓

2 - As analytic identities

Def $\mathbb{C}[[q]] := \left\{ \sum_{k=0}^{\infty} a_k q^k \;\middle|\; a_k \in \mathbb{C} \right\}$

- $\sum a_k q^k = \sum b_k q^k$ iff $a_k = b_k$ for all $k = 0, 1, \ldots$
- $\sum_{k=1}^{\infty} a_k q^k + \sum_{k=0}^{\infty} b_k q^k = \sum_{k=1}^{\infty} (a_k + b_k) q^k$
- $\sum_{k=1}^{\infty} a_k q^k \sum_{k=0}^{\infty} b_k q^k = \sum_{k=0}^{\infty} c_k q^k$

$c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_0 b_k$ ← Cauchy product

- associative
- commutative
- 0
- add inverse
- 1

- $c \sum a_k q^k = \sum c a_k q^k$ (scalar multiplication)

v.s. + ring = algebra.

Inverses We use geometric series

$$(1-q)^{-1} = \frac{1}{1-q} = 1 + q + q^2 + \cdots$$

proof: (FPS) $(1-q)(1+q+q^2+\cdots) \overset{?}{=} 1 + 0q + 0q^2 + \cdots$ ✓

Coeff of q^0 : $1 \cdot 1 = 1$

q : $-1 + 1 = 0$

q^2 : $-1 + 1 = 0$

$\vdots$

q^n : 0

} match the RHS

$\square$

In general $f(q)$, $f(0) \neq 0$,

$$\frac{1}{f(q)} = \frac{1}{a_0 + a_1 q + \cdots} \qquad a_0 \neq 0$$

$$= \frac{1}{a_0}\left(\frac{1}{1 - \frac{(-q)}{a_0}(a_1 + a_2 q + \cdots)} \right)$$

$$= \frac{1}{a_0}\left(1 + \frac{(-q)}{a_0}(a_1 + a_2 q + \cdots) + \frac{(-q)^2}{a_0^2}(a_1 + a_2 q + \cdots)^2 + \cdots \right)$$

Calculate coefficients of q^n is doable, a finite process for any n.

So $f(q)^{-1}$ exists in FPS (provided $f(0) \neq 0$).

$$\sum \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \sum_k q^{k^2}(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\cdots(1+q^k+q^{2k}+\cdots)$$

in principle, you can calculate q^n for any n.

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})} = \frac{1}{(1-q)(1-q^6)(1-q^{11})\cdots} \quad \frac{1}{(1-q^4)(1-q^9)\cdots}$$

$$= (1+q+q^2+\cdots)(1+q^6+\cdots)\quad\cdots$$

Both sides can be interpreted as FPS.

Use Sage : www.sagemath.org

exercises: FPS calculations can be done on Sage.

The simplest continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

$$1 = 1$$

$$1+\frac{1}{1} = 2$$

$$1+\cfrac{1}{1+\frac{1}{1}} = 1+\frac{1}{2} = 3/2$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}} = 1+\frac{1}{3/2} = 1+2/3 = 5/3$$

$$W_n = 1+\frac{1}{W_{n-1}} \rightarrow \frac{F_n}{F_{n-1}} = 1+\frac{F_{n-2}}{F_{n-1}}$$

$$W_n = \frac{F_n}{F_{n-1}}$$

$\Rightarrow F_n = F_{n-1}+F_{n-2}$ (3 term recurrence relation)

$F_0 = 1, F_1 = 1, 2, 3, 5, 8, 13, 21, \ldots$ Fibonacci Sequence

What did Ramanujan do

$$1+\cfrac{1}{1+\cfrac{1}{1+\cdots}} \longrightarrow 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{+\cdots}}}}$$

"q-analogues"

Simplest q-analogue $\quad 1+1+\cdots+1 = n$

$$1+q+\cdots+q^{n-1} = \frac{1-q^n}{1-q} \quad \leftarrow \text{Geometric sequence.}$$

$$1$$

$$1+q$$

$$1+\frac{q}{1+q^2} = \frac{1+q+q^2}{1+q^2}$$

$$1+\cfrac{q}{1+\cfrac{q^2}{1+q^3}}$$

$$C(z) = 1+\cfrac{qz}{1+\cfrac{q^2z}{1+\cfrac{q^3z}{1+\cdots}}}$$

$$C(z) = 1+\frac{q}{C(zq)} \quad -\textcircled{*}$$

Some more algebra suggests $C(z) = \dfrac{H(z)}{H(zq)}$

$$\frac{H(z)}{H(zq)} = 1+\frac{qzH(zq^2)}{H(zq)}$$

$$\Rightarrow H(z) = H(zq)+qzH(zq^2) \quad \textcircled{**}$$

Assume $H(z) = \sum_{k=0}^{\infty} a_k z^k$, plug in $\textcircled{**}$

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k q^k z^k + qz\sum_{k=0}^{\infty} a_k q^{2k} z^k$$

Compare coeff of z^k: $\quad a_k = a_k q^k + a_{k-1} q^{2k-2+1} \Rightarrow a_k(1-q^k) = a_{k-1} q^{2k-1}$

$$a_k(1-q^k) = a_{k-1}\, q^{2k-1}$$

$$a_k = a_{k-1}\frac{q^{2k-1}}{1-q^k}$$

$$= a_{k-2}\frac{q^{2k-1+2k-3}}{(1-q^k)(1-q^{k-1})}$$

$$= \vdots$$

$$= a_0 \frac{q^{1+3+5+\cdots+2k-1}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

$$= a_0 \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

$$H(z) = a_0 \sum_{k=1}^{\infty} \frac{q^{k^2} z^k}{(1-q)\cdots(1-q^k)}$$ Can take $a_0 = 1$

$$C(1) = \frac{H(1)}{H(q)} = \frac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(1-q)\cdots(1-q^k)}}{\displaystyle\sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(1-q)\cdots(1-q^k)}}$$

← Sum side of RR1

← Sum side of RR2

Askey → suggested Ramanujan got it this way.

Product side.

Recall proof of Geometric sum.

$$S = \underbrace{1+q}+q^2+\cdots+q^{n-1}$$

$$S(1-q) = 1-q+q(1-q)+\cdots+q^{n-1}(1-q)$$

$$= 1-\cancel{q}+\cancel{q}+ \text{higher powers of } q$$

$$= 1 + \text{higher powers of } q$$

Moral of story

$$S = 1 + q + \text{higher powers}$$

$$S(1-q) = 1 + \text{higher powers} \cdots$$

Consider $H(1) = 1 + \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1+q)(1-q^2)(1-q^3)} + \cdots$

$$= 1 + q + \text{higher powers}$$

$$H(1)(1-q) = 1(1-q) + q + \frac{q^4}{1-q^2} + \cdots$$

$$= 1 + \frac{q^4}{1-q^2} + \cdots$$

$$= 1 + q^4 + \text{higher powers}$$

$$H(1)(1-q)(1-q^4) = (1-q^4) + q^4(1+q^2) + \frac{q^9(1+q^2)}{1-q^3} + \cdots$$

$$= 1 + q^6 + \text{higher powers}$$

$$H(1)(1-q)(1-q^4)(1-q^6)\cdots = \text{(can hope that)} = 1$$

$$H(1)(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots$$

powers are 1, 4 mod 5

$$H(1) = \frac{1}{(1-q)(1-q^4)(1-q^6)\cdots}$$

Trick due to Euler.

This gives product side of RR1.

Ex: ① Work out Product side of RR2

② $\zeta(s) = \frac{1}{1} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$

do same trick to discover Euler's Product formula for zeta function

Some notation q-rising factorials

$$(A;q)_n = \begin{cases} 1 & \text{if } n=0 \\ (1-A)(1-Aq)\cdots(1-Aq^{n-1}) & , n>0 \end{cases}$$

$$(A;q)_\infty := \prod_{k=1}^{\infty}(1-Aq^k)$$ ← as FPS ✓ as analytical object convergence → $|q|<1$

RR I $$\sum_{k=1}^{\infty}\frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q;q^5)_\infty}\frac{1}{(q^4;q^5)_\infty}, \quad |q|<1$$

RR II $$\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

Goals

1. (Re) organize the Notebooks of Ramanujan / Berndt Ramanujan's Notebooks
   Andrews + Berndt Lost Notebook
2. Explain the identities / thms, if possible find how someone can discover.
3. Focus on techniques that work in many contexts
4. Give background where needed
   - ⇝ undergrad level
   - ⇝ graduate student
5. Written notes
6. Exercises so you can learn the techniques and gain familiarity with the results

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$$=\frac{1}{(1-q^2)(1-q^7)(1-q^{12})(1-q^{17})\cdots}\times\frac{1}{(1-q^3)(1-q^8)(1-q^{13})\cdots}$$

> “He worked, far more than a majority of modern mathematicians, by induction from numerical examples.”
>
> *–G. H. Hardy, about Ramanujan*

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