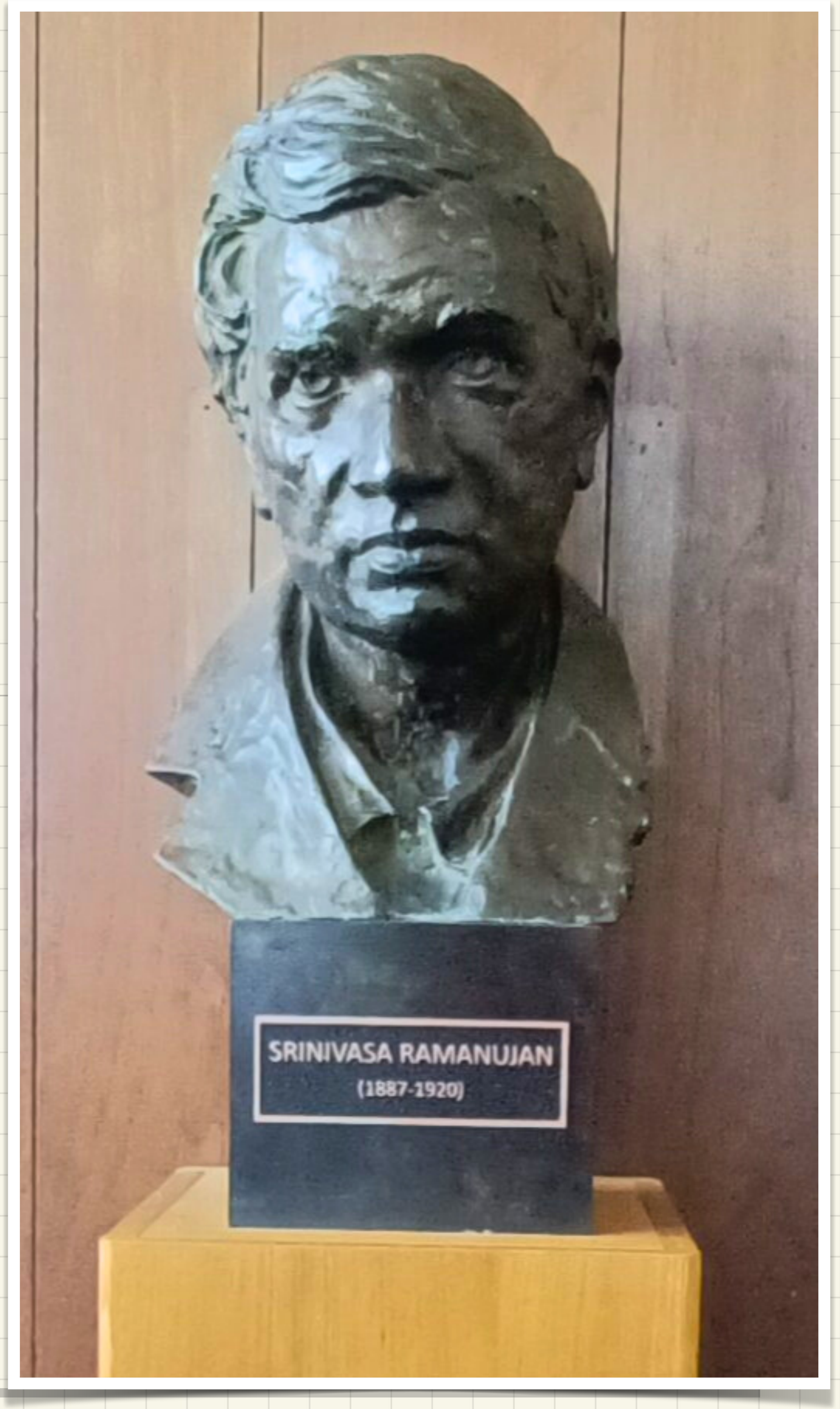
RAMANUJAN EXPLAINED

Lectures by Gaurav Bhatnagar

Lecture II: The q-binomial Theorem



THE q-BINOMIAL THEOREM

$$\frac{(-b;q)_\infty}{(a;q)_\infty} = \sum_{k=0}^{\infty} \frac{(-b/a;q)_k}{(q;q)_k} a^k$$

Notation : q-rising factorials

$(a;q)_0 := 1$

$(a;q)_k := \underbrace{(1-a)(1-aq)\cdots(1-aq^{k-1})}_{k\text{-terms}}$

$= \prod_{j=0}^{k-1}(1-aq^j)$ the "base" ↓ q

$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j)$ ← as a FPS in q or analytic function

Facts look at

1. $(z;q)_\infty$ as a function of q. For fixed $z \in \mathbb{C}$, product converges absolutely for $|q|<1$.
2. For fixed q, with $|q|<1$, we see $(z;q)_\infty$ as a function of z is entire ie analytic for all $z \in \mathbb{C}$.

Proof : deferred

Remarks

1. $$\frac{(a;q)_\infty}{(aq^k;q)_\infty} = \prod_{j=0}^{k-1}(1-aq^j) = (a;q)_k$$
if k is not a non-negative integer, we can use this to define q-rising factorials. (ie $k \in \mathbb{C}$).

2. $$\lim_{q\to 1}(a;q)_k = \lim_{q\to 1}\prod_{j=0}^{k-1}(1-aq^j) = (1-a)^k$$
So $$\lim_{q\to 1}\frac{(a;q)_\infty}{(aq^k;q)_\infty} = (1-a)^k$$

Ramanujan's III 16.1 (i)

$$\lim_{q\to 1}\frac{(q;q)_\infty}{(aq^x;q)_\infty} = (1-a)^x$$

3. For $|a|<1$, the RHS can be expanded using the binomial theorem

4. $$\lim_{q\to 1}\frac{(q^a;q)_k}{(1-q)^k} = \lim_{q\to 1}\frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})}{(1-q)(1-q)\cdots(1-q)}$$
$$= a(a+1)\cdots(a+k-1)$$
$$\parallel$$
$(a)_k$ ← rising factorial

Note $(1)_k = k!$

Example

$$E_q(z) := \sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}$$ "discovery" is an exercise in Ch 1 using Euler's trick

We show the sum converges for $|z|<1$ (provided $|q|<1$)

Let $t_k = \dfrac{z^k}{(q;q)_k}$

$$\left|\frac{t_{k+1}}{t_k}\right| = \left|\frac{z^{k+1}}{z^k}\,\frac{(1-q)\cdots(1-q^k)}{(1-q)\cdots(1-q^k)(1-q^{k+1})}\right|$$
$$= \left|z\,\frac{1}{1-q^{k+1}}\right| \to |z| \text{ when } k\to\infty$$ because $q^{k+1}\to 0$

By ratio test, $\sum t_k$ converges when $|z|<1$.

If we replace z by $z(1-q)$

$$E_q(z(1-q)) = \sum_{k=0}^{\infty}\frac{z^k(1-q)^k}{(q;q)_k}$$

$$\lim_{q\to 1}\frac{z^k(1-q)^k}{(q;q)_k} = \lim_{q\to 1}\frac{z^k(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)}$$
$$= \lim_{q\to 1}\frac{z^k}{k!}\qquad\left(\lim_{q\to 1}\frac{1-q^a}{1-q}=a\right)$$

$$E_q(z) \to \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z \quad (\text{termwise})$$

We say $E_q(z)$ is a q-analogue of e^z.
↳ there are more q-analogues

This is actually a special case of the q-binomial theorem

How to guess Newton's Binomial theorem

Finite form:

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$
↳ 'n choose k'
$$= \frac{n!}{k!(n-k)!}$$
= # of ways of choosing k-subsets of an n set

$n = 0,1,2,3,\ldots$ ✓

n is not a non-negative integer.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \binom{n}{k} = 0 \quad k > n$$

LHS makes sense if n is replaced by a – real / Complex #.

$(1+x)^a \to a \exp \lg(1+x)$

$$= \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

$$= \sum_{k=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k$$

$$\rightsquigarrow \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k$$

$$= (1+x)^a$$

The series converges for $|x|<1$.

Outline a proof in the exercises.

$$(1+x)^a = \sum_{k=0}^{\infty} \frac{(-a)(-a+1)\cdots(-a+k-1)}{k!} (-x)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-a)_k}{k!} (-x)^k.$$

q-binomial Thm

Thm Entry III.16.2. Let $|q|<1$, $|a|<1$. Then

$$\frac{(-b;q)_\infty}{(a;q)_\infty} = \sum_{k=0}^{\infty} \frac{(-b/a;q)_k}{(q;q)_k} a^k$$

Proof: (Ramanujan) Consider

$$F(a,b,z) = \frac{(bz;q)_\infty}{(az;q)_\infty} = \prod_{k=0}^{\infty} \frac{(1-bzq^k)}{(1-azq^k)}$$

expand as a FPS in z. $\quad = a_0 + a_1 z + a_2 z^2 + \cdots$

$$(1-az) F(a,b,z) = (1-bz) F(a,b,zq)$$

$$(1-az)(a_0 + a_1 z + \cdots) = (1-bz)(a_0 + a_1 zq + a_2 z^2 q^2 + \cdots)$$

Compare coefficients of z^k on both sides, $k>0$

$$a_k - a a_{k-1} = q^k a_k - b a_{k-1} q^{k-1}$$

$$a_k(1-q^k) = (a - bq^{k-1}) a_{k-1} = (1 - bq^{k-1}/a)\, a\, a_{k-1}$$

$$a_k = \frac{1-bq^{k-1}/a}{1-q^k}\, a\, a_{k-1}$$

$$= \frac{(1-bq^{k-1}/a)(1-bq^{k-2}/a)}{(1-q^k)(1-q^{k-1})} a^2 a_{k-2}$$

$$= \cdots$$

$$= \frac{(1-bq^{k-1}/a)\cdots(1-b/a)}{(1-q^k)\cdots(1-q)} a^k a_0$$

But $a_0 = 1$ (why?)

So we get (as FPS)

$$F(a,b,z) = \frac{(bz;q)_\infty}{(az;q)_\infty} = \sum_{k=0}^{\infty} \frac{(b/a;q)_k}{(q;q)_k} a^k z^k$$

The parameter z is not needed.

Take $a \to a/z$, $b \to -b/z$

$$\frac{(-b;q)_\infty}{(a;q)_\infty} = \sum_{k=0}^{\infty} \frac{(-b/a;q)_k}{(q;q)_k} a^k.$$

Remark: Proof works as analytic identity when $|a|<1$ (for series) and $|q|<1$ both products and series.

Usually q-binomial theorem is written as

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k$$

$|z|<1$, $|q|<1$.

provided denominators are not 0.

Fact: $(z;q)_\infty = 0$ only when $z = q^0, q^1, q^2, \ldots$

$$(z;q)_\infty = (1-z)(1-zq)\cdots$$

Proof (later).

Example: Suppose $a = q^{-n}$, $n \geq 0$.

$$\frac{(q^{-n}z;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k}{(q;q)_k} z^k$$

$$(q^{-n};q)_k = (1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+k-1})$$

$$= 0 \text{ when } k > n$$

Sum becomes a terminating sum

$$(1-q^{-n}z)(1-q^{-n+1}z)\cdots(1-zq^{-1}) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} z^k$$

Replace z by zq^n

$$(1-z)(1-zq)\cdots(1-zq^{n-1}) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} z^k q^{nk}$$

$$\overset{\text{Ex}}{=} \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} (-1)^k z^k$$

$q \to 1$

$$(1-z)^n = \sum_{k=0}^{n} \binom{n}{k} (-z)^k \leftarrow \text{Binomial theorem.}$$